Examples Computing $\mathcal{L}^{-1}\{F(s)\}$ using Formulas & Rules

Easy Examples

- $\mathcal{L}^{-1}\left\{\frac{s-1}{s^2-2s+5}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{s-1}{s^2-2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\}$$
$$= e^{t}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}$$
$$= \boxed{e^{t}\cos 2t}$$

- $\mathcal{L}^{-1}\left\{3/(s+2)^2\right\}$

$$\mathcal{L}^{-1}\left\{3/(s+2)^2\right\} = e^{-2t}\,\mathcal{L}^{-1}\left\{3/s^2\right\}$$
$$= e^{-2t}\cdot 3\;\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$
$$= \boxed{3e^{-2t}\cdot t}$$

- $\mathcal{L}^{-1}\left\{e^{-3s}\cdot\frac{2}{s^2+4}\right\}$

$$\mathcal{L}^{-1}\left\{e^{-3s}\frac{2}{s^2+4}\right\} = u_3(t)\,\ell^3\left[\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}\right]$$
$$= u_3(t)\,\ell^3\left[\sin 2t\right]$$
$$= \boxed{u_3(t)\cdot\sin 2(t-3)}$$

Medium Examples

- $\mathcal{L}^{-1}\left\{\frac{s}{s^2-2s+5}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s-1)^2+4}\right\} = e^{t}\,\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4}\right\}$$
$$= e^{t}\left(\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}\right)$$
$$= \boxed{e^{t}\left(\cos 2t + \tfrac{1}{2}\sin 2t\right)}$$

- $\mathcal{L}^{-1}\left\{\frac{s}{s^2-2s-3}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s-3)(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s-3}+\frac{B}{s+1}\right\}$$

$A = \frac{3}{3+1} = 3/4$

$B = \frac{-1}{-1-3} = 1/4$

$$= \mathcal{L}^{-1}\left\{3/4\,\frac{1}{s-3} + 1/4\,\frac{1}{s+1}\right\}$$
$$= 3/4\,\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + 1/4\,\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$
$$= \boxed{3/4\cdot e^{3t} + 1/4\cdot e^{-t}}$$

- $\mathcal{L}^{-1}\left\{\frac{s+4}{s(s^2+2)}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{s+4}{s(s^2+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s}+\frac{Bs+C}{s^2+2}\right\}$$

$A = \frac{0+4}{0+2} = \underline{\underline{2}}$

$A(s^2+2)+(Bs+C)s = s+4$

$(s^2)\quad A+B=0 \leadsto B=\underline{\underline{-2}}$

$(s)\quad 0+C=1 \leadsto C=\underline{\underline{1}}$

$(1)\quad 2A+0=4$

$$= \mathcal{L}^{-1}\left\{\frac{2}{s}+\frac{-2s+1}{s^2+2}\right\}$$
$$= 2\mathcal{L}^{-1}\{1/s\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\right\}$$
$$= \boxed{2 - 2\cos\sqrt{2}t + 1/\sqrt{2}\,\sin\sqrt{2}t}$$

• $\mathcal{L}^{-1}\left\{\frac{2s^2+3s-1}{(s-1)^3}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{2s^2+3s-1}{(s-1)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{(s-1)^3}\right\}$$

$A(s-1)^2+B(s-1)+C = 2s^2+3s-1$

$(s^2)\quad A+0+0=2 \rightsquigarrow \underline{\underline{A=2}}$

$(s)\quad -2A+B+0=3 \rightsquigarrow \underline{\underline{B=7}}$

$(1)\quad A-B+C=-1 \rightsquigarrow \underline{\underline{C=4}}$

$$\Rightarrow = \mathcal{L}^{-1}\left\{\frac{2}{(s-1)}+\frac{7}{(s-1)^2}+\frac{4}{(s-1)^3}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}+7\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}+4\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3}\right\}$$

$$= \boxed{2e^t+7te^t+\frac{4}{2}t^2e^t}$$

• $\mathcal{L}^{-1}\left\{\frac{s-1}{s^2-2s+5}\cdot e^{-3s}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\,e^{-3s}\right\} = u_3(t)\,\ell^3\left[\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\}\right]$$

$$= u_3(t)\,\ell^3\left[e^t\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}\right]$$

$$= u_3(t)\,e^{(t-3)}\,\ell^3[\cos 2t]$$

$$= \boxed{u_3(t)\cdot e^{(t-3)}\cdot\cos 2(t-3)}$$

• $\mathcal{L}^{-1}\left\{\frac{3}{(s-2)^2}\cdot e^{-3s}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{3}{(s-2)^2}\,e^{-3s}\right\} = u_3(t)\,\ell^3\left[\mathcal{L}^{-1}\left\{\frac{3}{(s-2)^2}\right\}\right]$$

$$= u_3(t)\,\ell^3\left[e^{2t}\mathcal{L}^{-1}\{3/s^2\}\right]$$

$$= u_3(t)\cdot e^{2(t-3)}\,\ell^3[3t] = \boxed{u_3(t)\cdot e^{2(t-3)}\cdot 3(t-3)}$$

• $\mathcal{L}^{-1}\left\{\frac{2s^3+3s^2-1}{s^3}\cdot e^{-5s}\right\}$

$$= \mathcal{L}^{-1}\left\{\left(2+\frac{3}{s}-\frac{1}{s^3}\right)e^{-5s}\right\}$$

$$= \mathcal{L}^{-1}\left\{2e^{-5s}+\frac{3}{s}e^{-5s}-\frac{1}{s^3}e^{-5s}\right\}$$

$$= 2\mathcal{L}^{-1}\{e^{-5s}\}+3\mathcal{L}^{-1}\left\{\frac{1}{s}e^{-5s}\right\}-\mathcal{L}^{-1}\left\{1/s^3\,e^{-5s}\right\}$$

$$= 2\,\delta_5(t)+3\,u_5(t)\,\ell^5\left[\mathcal{L}^{-1}\{1/s\}\right]-u_5(t)\,\ell^5\left[\mathcal{L}^{-1}\{1/s^3\}\right]$$

$$= 2\,\delta_5(t)+3\,u_5(t)\,\ell^5[1]-u_5(t)\,\ell^5\left[\tfrac{1}{2}t^2\right]$$

$$= 2\,\delta_5(t)+3\,u_5(t)\cdot 1-u_5(t)\cdot\tfrac{1}{2}(t-5)^2$$

$$= \boxed{2\,\delta_5(t)+u_5(t)\cdot\left(3-\tfrac{1}{2}(t-5)^2\right)}$$

(Impulse function $\delta_5(t)$ is sometimes written $\delta(t-5)$)

Hard Examples

• $\mathcal{L}^{-1}\{1\}$

$$\mathcal{L}^{-1}\{1\} = \mathcal{L}^{-1}\{e^0\}$$

$$= \mathcal{L}^{-1}\{e^{0\cdot s}\}$$

$$= \delta_0(t)$$ (aka. "$\delta(t)$")

(Impulse at time $t=0$.)

(Sometimes we put this problem on the exam, and all of the professors laugh...)

• $\mathcal{L}^{-1}\left\{\frac{s+2}{s^2-2s+5}\, e^{-3s+2}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{s+2}{(s-1)^2+4}\, e^{-3s} e^{2}\right\} = e^2 \mathcal{L}^{-1}\left\{\frac{s+2}{(s-1)^2+4}\, e^{-3s}\right\}$$

$$= e^2 u_3(t)\, \ell^3\left[\mathcal{L}^{-1}\left\{\frac{s+2}{(s-1)^2+4}\right\}\right]$$

$$= e^2 u_3(t)\, \ell^3\left[\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4} + \frac{3}{(s-1)^2+4}\right\}\right]$$

$$= e^2 u_3(t)\, \ell^3\left[e^t \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + e^t\, \tfrac{3}{2}\, \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}\right]$$

$$= e^2 u_3(t)\, \ell^3\left[e^t \cos 2t + \tfrac{3}{2} e^t \sin 2t\right]$$

$$= e^2 u_3(t)\left(e^{t-3}\cos 2(t-3) + \tfrac{3}{2} e^{t-3} \sin 2(t-3)\right)$$

$$= \boxed{u_3(t)\, e^{t-1}\left(\cos 2(t-3) + \tfrac{3}{2}\sin 2(t-3)\right)}$$

• $\mathcal{L}^{-1}\left\{\frac{2s-3}{s^2-3s+2}\, e^{-4s+2}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{2s-3}{(s-2)(s-1)}\, e^{-4s} e^{2}\right\} = e^2 \mathcal{L}^{-1}\left\{\frac{2s-3}{(s-2)(s-1)}\, e^{-4s}\right\}$$

$$= e^2 u_4(t)\, \ell^4\left[\mathcal{L}^{-1}\left\{\frac{A}{s-2} + \frac{B}{s-1}\right\}\right]$$

$$= e^2 u_4(t)\, \ell^4\left[\mathcal{L}^{-1}\left\{\frac{1}{s-2} + \frac{1}{s-1}\right\}\right]$$

Partial Fractions

$A(s-1) + B(s-2) = 2s-3$

$s=2 \Rightarrow A = \frac{4-3}{2-1} = 1$

$s=1 \Rightarrow B = \frac{2-3}{1-2} = 1$

$$= e^2 u_4(t)\, \ell^4\left[e^{2t} + e^t\right]$$

$$= e^2 u_4(t)\left(e^{2(t-4)} + e^{t-4}\right)$$

$$= \boxed{u_4(t)\cdot\left(e^{2(t-4)+2} + e^{(t-4)+2}\right)}$$

• $\mathcal{L}^{-1}\left\{\frac{s^3}{(s+2)^3}\, e^{-5s}\right\}$

[Because the numerator s^3 has same degree as denominator $(s+2)^3$, the answer will have an impulse function. So pulling out the e^{-5s} will have a strange effect... Watch closely!]

$$\mathcal{L}^{-1}\left\{\frac{s^3}{(s+2)^3}\, e^{-5s}\right\} = u_5(t)\, \ell^5\left[\mathcal{L}^{-1}\left\{\frac{s^3}{(s+2)^3}\right\}\right]$$

$$= u_5(t)\, \ell^5\left[e^{-2t}\mathcal{L}^{-1}\left\{\frac{(s-2)^3}{s^3}\right\}\right]$$

(changing $s+2 \mapsto s$ makes $s \mapsto s-2$)

$$= u_5(t)\, \ell^5\left[e^{-2t}\mathcal{L}^{-1}\left\{\frac{s^3-6s^2+12s-8}{s^3}\right\}\right]$$

$$= u_5(t)\, \ell^5\left[e^{-2t}\mathcal{L}^{-1}\left\{1 - \frac{6}{s} + \frac{12}{s^2} - \frac{8}{s^3}\right\}\right]$$

$$= u_5(t)\, \ell^5\left[e^{-2t}\left(\mathcal{L}^{-1}\{1\} - 6\mathcal{L}^{-1}\left\{\tfrac{1}{s}\right\} + 12\mathcal{L}^{-1}\left\{\tfrac{1}{s^2}\right\} - \tfrac{8}{2}\mathcal{L}^{-1}\left\{\tfrac{2}{s^3}\right\}\right)\right]$$

$$= u_5(t)\, \ell^5\left[e^{-2t}\left(\delta(t) - 6 + 12t - \tfrac{8}{2}t^2\right)\right]$$

$$= u_5(t)\, \ell^5\left[\delta(t) + e^{-2t}\left(-6 + 12t - \tfrac{8}{2}t^2\right)\right]$$

(**Note**: $e^{-2t}\delta(t) = e^{-2\cdot 0}\delta(t) = 1\cdot\delta(t) = \delta(t)$)

$$= \boxed{\delta(t-5) + u_5(t)\, e^{-2(t-5)}\left(-6 + 12(t-5) - \tfrac{8}{2}(t-5)^2\right)}$$

(**Note**: $u_5(t)\, \ell^5[\delta(t)] = u_5(t)\,\delta(t-5) = \delta(t-5)$)

Three Basic Rational Function Types (with Exp. and Step)

- $\mathcal{L}^{-1}\left\{e^{-5s}\cdot\frac{s}{s^2+8s+25}\right\}$

$$\mathcal{L}^{-1}\left\{e^{-5s}\frac{s}{(s+4)^2+9}\right\} = u_5(t)\,\ell^{5}\left[\mathcal{L}^{-1}\left\{\frac{s}{(s+4)^2+9}\right\}\right]$$

$$= u_5(t)\,\ell^{5}\left[e^{-4t}\mathcal{L}^{-1}\left\{\frac{s-4}{s^2+9}\right\}\right]$$

$$= u_5(t)\,\ell^{5}\left[e^{-4t}\left(\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} - \tfrac{4}{3}\mathcal{L}^{-1}\left\{\frac{1\cdot 3}{s^2+9}\right\}\right)\right]$$

$$= u_5(t)\,\ell^{5}\left[e^{-4t}\left(\cos 3t - \tfrac{4}{3}\cdot\sin 3t\right)\right]$$

$$= \boxed{u_5(t)\,e^{-4(t-5)}\left(\cos 3(t-5) - \tfrac{4}{3}\sin 3(t-5)\right)}$$

- $\mathcal{L}^{-1}\left\{e^{-5s}\cdot\frac{s}{s^2+8s+16}\right\}$

$$\mathcal{L}^{-1}\left\{e^{-5s}\frac{s}{(s+4)^2}\right\} = u_5(t)\,\ell^{5}\left[\mathcal{L}^{-1}\left\{\frac{s}{(s+4)^2}\right\}\right]$$

$$= u_5(t)\,\ell^{5}\left[e^{-4t}\mathcal{L}^{-1}\left\{\frac{s-4}{s^2}\right\}\right]$$

$$= u_5(t)\,\ell^{5}\left[e^{-4t}\left(\mathcal{L}^{-1}\left\{\tfrac{1}{s}\right\} - 4\mathcal{L}^{-1}\left\{\tfrac{1}{s^2}\right\}\right)\right]$$

$$= u_5(t)\,\ell^{5}\left[e^{-4t}\left(1 - 4t\right)\right]$$

$$= \boxed{u_5(t)\,e^{-4(t-5)}\left(1 - 4(t-5)\right)}$$

- $\mathcal{L}^{-1}\left\{e^{-5s}\cdot\frac{s}{s^2+8s+12}\right\}$

$$\mathcal{L}^{-1}\left\{e^{-5s}\frac{s}{(s+2)(s+6)}\right\} = u_5(t)\,\ell^{5}\left[\mathcal{L}^{-1}\left\{\frac{s}{(s+2)(s+6)}\right\}\right]$$

$$= u_5(t)\,\ell^{5}\left[\mathcal{L}^{-1}\left\{\frac{A}{s+2} + \frac{B}{s+6}\right\}\right]$$

Partial Fractions

$A(s+6) + B(s+2) = s$

$s=-2$: $A(4) + B(0) = -2$
$\rightarrow A = -\tfrac{1}{2}$

$s=-6$: $A(0) + B(-4) = -6$
$\rightarrow B = \tfrac{3}{2}$

$$\rightarrow = u_5(t)\,\ell^{5}\left[\mathcal{L}^{-1}\left\{-\tfrac{1}{2}\frac{1}{s+2} + \tfrac{3}{2}\frac{1}{s+6}\right\}\right]$$

$$= u_5(t)\,\ell^{5}\left[-\tfrac{1}{2}e^{-2t} + \tfrac{3}{2}e^{-6t}\right]$$

$$= \boxed{u_5(t)\left(-\tfrac{1}{2}e^{-2(t-5)} + \tfrac{3}{2}e^{-6(t-5)}\right)}$$

Remark: When exponentials cross $\mathcal{L}$ and $\mathcal{L}^{-1}$ they become lags:

$$\mathcal{L}\left\{e^{-2t}t^3\right\} = \ell^{-2}\left[\mathcal{L}\{t^3\}\right]$$

$$\mathcal{L}^{-1}\left\{e^{-2s}\tfrac{1}{s^3}\right\} = u_2(t)\,\ell^{2}\left[\mathcal{L}^{-1}\left\{\tfrac{1}{s^3}\right\}\right]$$

Note: signs switch when crossing $\mathcal{L}^{-1}$

$\rightarrow$ In reverse, shifts become exponentials:

$$\mathcal{L}\left\{u_2(t)\,(t-2)^3\right\} = e^{-2s}\mathcal{L}\{t^3\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^3}\right\} = e^{2t}\mathcal{L}^{-1}\left\{\tfrac{1}{s^3}\right\}$$

Note: signs switch when crossing $\mathcal{L}^{-1}$

(Also, lags in time domain are always paired with step functions $u_c(t)\,\ell^{c}[\cdots]$ because in time domain $\mathcal{L}$ assumes nothing exists before time $t=0$.)